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SENSITIVITY REDUCTION OVER A FREQUENCY BAND(U)

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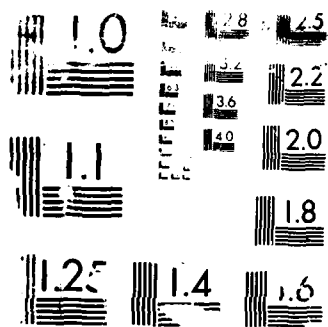
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SENSITIVITY REDUCTION OVER A FREQUENCY BAND

by

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ABSTRACT

This paper considers the problem of reducing the sensitivity of a possibly infinite dimensional linear single-input single-output system over a finite frequency interval by feedback. Specifically the following are proven: (i) if one wants to bound the overall sensitivity, the existence of a nontrivial inner part inhibits the reduction of the sensitivity over the interval: (ii) in a system that is continuous and has at most countably many zeros on the imaginary axis, one can reduce the sensitivity over the interval arbitrarily small while the overall sensitivity is kept bounded if and only if the system is outer and has no zeros on the interval. These extend results for rational transfer functions.

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I. INTRODUCTION

This paper considers the problem of reducing the sensitivity of a linear single-input single-output system over a finite frequency interval by feedback.

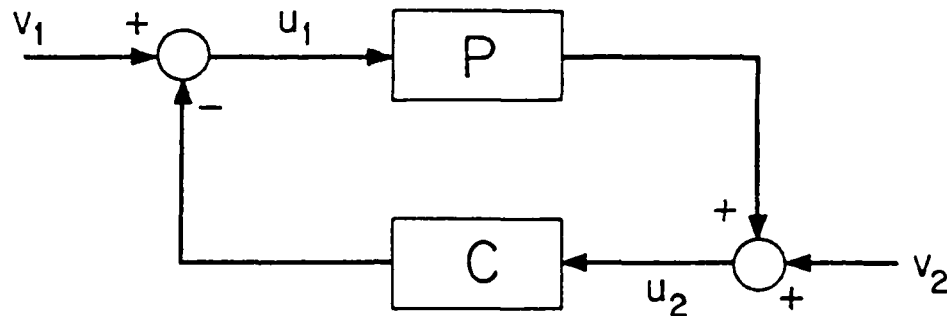


Fig. 1

The feedback system is described by Fig. 1. P is a given system and we assume $P \in H^\infty$ (i.e., stable) and C is a feedback. We say that the feedback stabilizes the system if the transfer functions from (v_1, v_2) to (u_1, u_2) all belong to H^∞ .

The closed loop sensitivity S is the transfer function from v_1 to u_2 and is given by

$$S(s) = (1 + P(s)C(s))^{-1} \quad (1.1)$$

The problem of sensitivity reduction over a frequency band X is stated as follows. Let x be the characteristic function of a given bounded set $X \subset (-\infty, \infty)$, on the imaginary axis, i.e.,

$$x(j\omega) = \begin{cases} 1 & \text{if } \omega \in X, \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

For given $\varepsilon > 0$ and $M > 1$, find a stabilizing feedback for which the sensitivity satisfies

$$||xS||_{\infty} < \varepsilon, \quad ||S||_{\infty} < M. \quad (1.3)$$

Here are our main results.

Theorem 1.

Suppose $P \in H^{\infty}$ has a nontrivial inner part and x is the characteristic function of a subset of the imaginary axis which has positive measure. Then

$$\inf_{||S||_{\infty} < M} ||xS||_{\infty} > 0,$$

where $M > 1$ and the infimum is taken over all stabilizing compensators.

Theorem 2.

Suppose $P \in H^{\infty}$ is continuous and has at most countably many zeros on the imaginary axis. Let x be the characterization function of a compact set $X \subset (-\infty, \infty)$ on the imaginary axis. Then for any $\varepsilon > 0$ and any $M > 1$ there exist a stabilizing compensator such that

$$||xS||_{\infty} < \varepsilon, \quad ||S||_{\infty} < M,$$

if and only if P is outer and has no zeros on jX .



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Previous discussions of this problem appear in [1] - [5]. In [1], it was shown that if the plant P is analytic, is bounded, has no zero in $\operatorname{Re} s \geq 0$, and satisfies an attenuation condition at $s = \infty$, then for any $\epsilon > 0$ and $M > 1$ the problem has a solution. Especially the problem is solvable when P is of minimum phase. Theorem 2 generalizes this result, and seems to illuminate more on the structural aspects of the sensitivity reduction problem. In [3], in the framework of rational plants, it was shown that if the plant P has a right half plane zero then there exists a positive number k such that

$$\|xS\|_{\infty} \geq \|S\|_{\infty}^k.$$

Hence given $M > 1$, there is $\epsilon > 0$ such that the problem has no solution. Theorem 1 is a natural extension of this statement. In [5], it was shown that if the plant is analytic and has no zero in some region containing $\operatorname{Re} s \geq 0$, and satisfies some intricate condition near $s = \infty$, then for any $\epsilon > 0$ and $M > 2+L$ (L is determined by the condition) the problem has a solution. However the condition seems rather difficult to check. The difficulty was demonstrated by the authors' wrong conclusion that for $P(s) = e^{-s}/(s+1)$ (which has a nontrivial inner part), and some $M > 2$, the problem has a solution for any $\epsilon > 0$.

II. PRELIMINARIES AND NOTATIONS

2.1 Parametrization of Stabilizing Feedbacks

We parametrize feedbacks achieving stability. The parametrization was introduced in [6] and modified in [7]. The following is a corollary of [8] for a stable system.

Proposition 3:

Assume P is stable ($P \in H^\infty$). Then a feedback C stabilizes the system if and only if there exists $h \in H^\infty$, $Ph \neq 1$ such that

$$C = \frac{h}{1-Ph}. \quad (2.1)$$

Substituting (2.1) to (1.1), we have

$$S = 1-Ph. \quad (2.2)$$

Therefore our problem is reduced to that of finding $h \in H^\infty$ satisfying

$$\|x(1-Ph)\|_\infty < \varepsilon, \quad \|1-Ph\|_\infty < M \quad (2.3)$$

for given $\varepsilon > 0$, $M > 1$.

2.2 H^p Functions

H^2 and H^∞ are the Hardy spaces of analytic functions on the right half

plane with L^2 and L^∞ boundary values, respectively. [9], [10] are good sources on H^p spaces, inner-outer factorizations, etc. The following is from [10] and is worthy of note.

Proposition 4 [10].

Assume $P \in H^\infty$ and let $K = H^2 \ominus PH^2$ (or $K = (PH^2)^\perp$). Then $K = \{0\}$ if and only if P is outer.

2.3 σ -Inner Product and σ -Norm

The Laplace transformation L defines an isometric isomorphism from $L^2[0, \infty)$ to H^2 . We shall use both the time domain and the frequency domain in our analysis.

We denote the usual inner product and norm of H^2 (respectively, $L^2[0, \infty)$) by $\langle \cdot, \cdot \rangle$ and $||\cdot||$. For future use we introduce also a whole family of additional inner products and norms as follows. Given $\sigma > 0$, and $f, g \in H^2$, define the σ -inner product and the σ -norm by

$$\langle f, g \rangle_\sigma = (2\pi)^{-1} \int_{-\infty}^{\infty} f(\sigma + j\omega) \overline{g(\sigma + j\omega)} d\omega \quad (2.4)$$

$$||f||_\sigma = \langle f, f \rangle_\sigma^{1/2} \quad (2.5)$$

Since L is an isometry, there hold

$$\langle f, g \rangle_\sigma = \int_0^\infty L^{-1}(f)(t) \overline{L^{-1}(g)(t)} e^{-2\sigma t} dt \quad (2.6)$$

$$\|f\|_\sigma = \left(\int_0^\infty |L^{-1}(f)(t)|^2 e^{-2\sigma t} dt \right)^{1/2} \quad (2.7)$$

If $L^{-1}(f)$ and $L^{-1}(g)$ are supported within the compact interval $[0, T]$, given some $T > 0$, then

$$e^{-\sigma T} \|f\| \leq \|f\|_\sigma \leq \|f\| \quad (2.8)$$

and

$$|\langle f, g \rangle - \langle f, g \rangle_\sigma| \leq (1 - e^{-2\sigma T}) (\|f\| + \|g\|)^2 \quad (2.9)$$

For $x \in L^2[0, \infty)$, define $x_T \in L^2[0, \infty)$ to be the truncation of x at time T , $T > 0$, i.e.

$$x_T(t) = \begin{cases} x(t) & t \leq T \\ 0 & \text{otherwise} \end{cases} \quad (2.10)$$

For $f = L(x)$, we denote $f_T = L(x_T)$. Notice that $x_T \rightarrow x$ and $f_T \rightarrow f$ as $T \rightarrow \infty$, in the usual topology of $L^2[0, \infty)$ and H^2 .

III. PROOFS OF THEOREMS 1 AND 2.

In proving Theorem 1 we use the following observations.

Lemma 5.

Let $\{g_n\} \subset H^\infty$ be a sequence such that $\|g_n\|_\infty < M$. Let x be the characteristic function of a set $X \subset [-\infty, \infty]$ of positive measure on the imaginary axis. Suppose $\|xg_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Then for any compact set Y in the open right half plane, $|g_n(s)| \rightarrow 0$, uniformly for $s \in Y$.

Proof: It seems convenient to establish the lemma in the disc. For $g \in H^\infty$, define

$$g_D(z) = g\left(\frac{1+z}{1-z}\right) \quad (3.1)$$

Then $g_D \in H^\infty(D)$ and $\|g\|_\infty = \|g_D\|_\infty$, where D is the unit disc on the complex plane. Let $Y_D = \{z | (1+z)/(1-z) \in Y\}$, and $X_D = \{z | (1+z)/(1-z) \in X\}$ be the inverse image of Y and X by the Mobius transformation respectively, and x_D be the characteristic function of X_D . Since Y is a compact set in the open right half plane, $Y_D \subset B(0, r)$ (\equiv the closed disc of radius r) for some $0 < r < 1$. By Jensen's inequality, we have

$$\log |g_D(z)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |g_D(e^{i\theta})| \operatorname{Re} \left[\frac{e^{i\theta} + z}{e^{i\theta} - z} \right] d\theta. \quad (3.2)$$

Note that for $z \in Y_D$ and $\theta \in [-\pi, \pi]$

$$\frac{1+r}{1-r} \geq \frac{1+|z|}{1-|z|} \geq \operatorname{Re} \left[\frac{e^{i\theta+z}}{e^{i\theta-z}} \right] \geq \frac{1-|z|}{1+|z|} \geq \frac{1-r}{1+r} \quad (3.3)$$

Using the inequalities (3.3), for $z \in Y_D$ we can find a uniform upper bound for the right hand side of (3.2).

$$\begin{aligned} & \log |g_D(z)| \\ & \leq \frac{1}{2\pi} \left[\frac{1+r}{1-r} \int_{-\pi}^{\pi} \log^+ |g_D(e^{i\theta})| d\theta - \frac{1-r}{1+r} \int_{-\pi}^{\pi} \log^+ |g_D(e^{i\theta})|^{-1} d\theta \right] \\ & \leq \frac{1}{2\pi} \left[\frac{1+r}{1-r} \int_{-\pi}^{\pi} |g_D(e^{i\theta})| d\theta - \frac{1-r}{1+r} \int_{X_D} \log^+ |g_D e^{i\theta}|^{-1} d\theta \right] \\ & \leq \frac{1+r}{1-r} \|g_D\|_{\infty} - \frac{1}{2\pi} \frac{1-r}{1+r} \log^+ (\|x_D g_D\|_{\infty}^{-1}) \mu(X_D) \end{aligned} \quad (3.4)$$

where

$$\log^+ x = \log\{\max(1, x)\} \geq 0, \quad (3.5)$$

and μ is the Lebesgue measure on the unit circle.

Let $\{g_{nD}\}$ be the sequence in $H^{\infty}(D)$ obtained from $\{g_n\}$ by the transformation (3.1). Note that

$$\|g_{nD}\|_{\infty} < M, \text{ and } \|x_D g_{nD}\|_{\infty} \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (3.6)$$

Applying (3.4) to the sequence $\{g_{nD}\}$, we see that the right hand side of

(3.4) tends to $-\infty$ uniformly for $z \in Y_D$. Hence $|g_n(s)| \rightarrow 0$ uniformly for $s \in Y$.

Q.E.D.

Corollary 6. Let $\{g_n\}$ be as in Lemma 5, $f \in H^2$ and $\sigma > 0$. Then $\|g_n f\|_\sigma \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Fix $\varepsilon > 0$. Since $f \in H^2$, there exists $\Omega > 0$ such that

$$(2\pi)^{-1} \int_{|\omega| > \Omega} |f(\sigma + j\omega)|^2 d\omega < \frac{\varepsilon}{2M^2} \quad (3.7)$$

This implies that

$$(2\pi)^{-1} \int_{|\omega| > \Omega} |g_n f(\sigma + j\omega)|^2 d\omega < \frac{\varepsilon}{2} \quad (3.8)$$

for all n , since $\|g_n\|_\infty < M$.

Applying Lemma 5 to the sequence $\{g_n\} \subset H^\infty$ and $Y = \{s | s = \sigma + j\omega, |\omega| \leq \Omega\}$, and Lebesgue's dominant convergence theorem, we see that

$$(2\pi)^{-1} \int_{|\omega| \leq \Omega} |g_n f(\sigma + j\omega)|^2 d\omega \rightarrow 0 \text{ as } n \rightarrow \infty \quad (3.9)$$

Thus for n large enough,

$$\|g_n f\|_\sigma^2 = (2\pi)^{-1} \int_{-\infty}^{\infty} |g_n f(\sigma + j\omega)|^2 d\omega < \varepsilon \quad (3.10)$$

This implies that $\|g_n f\|_\sigma \rightarrow 0$ as $n \rightarrow \infty$.

Q.E.D.

Claim 7. Let $\{g_n\} \subset H^\infty$, $\|g_n\|_\infty < M$ and $f \in H^2$. Then for each $\lambda > 0$, there exist some $T > 0$ and $\sigma > 0$ such that $|\langle f, g_n f \rangle - \langle f_T, g_n f \rangle_\sigma| < \lambda$ for all n .

Proof

$$\begin{aligned}
 & |\langle f, g_n f \rangle - \langle f_T, g_n f \rangle| \\
 &= |\langle f - f_T, g_n f \rangle| \\
 &\leq \|f - f_T\| \|g_n f\| \\
 &\leq \|g_n\|_\infty \|f - f_T\| \|f\| \\
 &\leq M \|f - f_T\| \|f\|
 \end{aligned} \tag{3.11}$$

Utilizing (2.9), we also have

$$\begin{aligned}
 & |\langle f_T, g_n f \rangle - \langle f_T, g_n f \rangle_\sigma| \\
 &= |\langle f_T, (g_n f)_T \rangle - \langle f_T, (g_n f)_T \rangle_\sigma| \\
 &\leq (1 - e^{-2\sigma T}) [\|f_T\| + \|(g_n f)_T\|]^2 \\
 &\leq (1 - e^{-2\sigma T}) [\|f\| + \|g_n f\|]^2 \\
 &\leq (1 + M)^2 (1 - e^{-2\sigma T}) \|f\|^2
 \end{aligned} \tag{3.12}$$

Recall that $f_T \rightarrow f$ as $T \rightarrow \infty$. Choose T sufficiently large so that $\|f - f_T\| < \lambda/2M\|f\|$. Then choose σ sufficiently small so that $(1 - e^{-2\sigma T}) < \lambda/2(1+M)^2\|f\|^2$. Combining (3.11) and (3.12), we have the desired inequality. Q.E.D.

For ease of reference, we repeat our results again.

Theorem 1.

Suppose $P \in H^\infty$ has a nontrivial inner part and x is the characteristic function of a subset of the imaginary axis which has positive measure. Then

$$\inf_{\|S\|_\infty < M} \|xS\|_\infty > 0, \quad (3.13)$$

where $M > 1$ and the infimum is taken over all stabilizing compensators.

Proof of Theorem 1.

On the contrary, assume that there exists a sequence $\{S_n\}$ of sensitivity functions, $S_n = 1 - Ph_n$, $h_n \in H^\infty$ with $\|S_n\|_\infty < M$, $\|xS_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$.

Set $K = H^2 \ominus PH^2$. The subspace K is nontrivial ($K \neq \{0\}$), by Proposition 4, since P has a nontrivial inner part. For any $f \in K$ and any $g \in H^2$, we have

$$\langle f, S_n g \rangle = \langle f, (1 - P_n) g \rangle \quad (3.14)$$

$$= \langle f, g \rangle - \langle f, P_n g \rangle$$

$$= \langle f, g \rangle;$$

in particular, for $f \in K$, $f \neq 0$,

$$\langle f, S_n f \rangle = \|f\|^2 > 0. \quad (3.15)$$

Hence in view of Claim 7, there exist $T > 0$, $\sigma > 0$, and $\delta > 0$ such that

$$\langle f_T, S_n f \rangle_\sigma > \delta \quad (3.16)$$

for all n .

On the other hand, by corollary 6

$$|\langle f_T, S_n f \rangle_\sigma| \leq \|f_T\|_\sigma \|S_n f\|_\sigma \rightarrow 0 \text{ as } n \rightarrow \infty \quad (3.17)$$

which is a contradiction.

Q.E.D.

Theorem 2.

Suppose $P \in H^\infty$ is continuous and has at most countably many zeros on the imaginary axis. Let x be the characterization function of a compact set jX ; $X \subset (-\infty, \infty)$ on the imaginary axis. Then for any $1 > \varepsilon > 0$ and any $M > 1$ there exists a stabilizing compensator such that

$$\|xS\|_{\infty} < \varepsilon, \quad \|S\|_{\infty} < M \quad (3.18)$$

if and only if P is outer and has no zeros on jX .

Proof: (Necessity) If P is not outer, then the conclusion follows from Theorem 1. Let P have a zero on X . Suppose then that there exists $h \in H^{\infty}$ such that

$$\|x(1-Ph)\|_{\infty} < \varepsilon, \quad \|1-Ph\|_{\infty} < M \quad (3.19)$$

Fix $\delta > 0$. Because P is continuous on the imaginary axis,

$$\mu[X \cap \{\omega \mid |P(i\omega)| < \delta \|h\|^{-1}\}] > 0, \text{ i.e., } \mu[X \cap \{\omega \mid |P(j\omega)h(j\omega)| < \delta\}] > 0$$

where μ is the Lebesgue measure on the imaginary axis. Since δ was arbitrary $\|x(1-Ph)\|_{\infty} \geq 1$, a contradiction.

(Sufficiency). The proof is by construction of $h \in H^{\infty}$ such that

$$\|x(1-Ph)\|_{\infty} < \varepsilon, \quad \|1-Ph\|_{\infty} < M \quad (3.20)$$

for given $0 < \varepsilon < 1$, $M > 1$.

Let

$$U = \{u: u = \infty \text{ or } P(ju) = 0\}. \quad (3.21)$$

From the assumption U is at most countable and $U \cap X = \emptyset$. Let $U = \{u_n, n=1,2,\dots\}$ be an enumeration of U .

Define $r_{n,a}$ by

$$r_{n,a}(s) = \begin{cases} \left[\frac{a(s - ju_n)}{a(s - ju_n) + 1} \right]^{2^{-(n+1)}} & u_n \neq \infty \\ \left[\frac{a}{s+a} \right]^{2^{-(n+1)}} & u_n = \infty, \end{cases} \quad (3.22)$$

where $a > 0$ is a parameter to be fixed later, and the branch of the $2^{-(n+1)}$ th complex root is decided in such a way that the positive real line is mapped by $r_{n,a}$ into itself. Eq. (3.22) defines an analytic function on the open right half plane, since the function $a(s - ju_n)/[a(s - ju_n) + 1]$ (or $a/(s+a)$) maps the open right half plane into itself. Furthermore, the following properties of $r_{n,a}$ are easily proved: (i) $r_{n,a} \in H^\infty$, $\|r_{n,a}\|_\infty = 1$; (ii) $r_{n,a}$ is outer; (iii) $r_{n,a}$ is continuous on the imaginary axis including ∞ ; (iv) $r_{n,a}(u_n) = 0$; (v) $\|x(1 - r_{n,a})\|_\infty \rightarrow 0$ as $a \rightarrow \infty$ (note that $X \cap U = \emptyset$); and (vi) $|\arg r_{n,a}(s)| \leq \pi 2^{-(n+2)}$, for any $s \in \{\operatorname{Re} s \geq 0\} \cup \{\infty\}$.

Given $\varepsilon > 0$ (from (3.17)), we choose $\eta > 0$ such that $|\log z| < \eta$, $z \in C$,

implies

$$|z-1| < \epsilon \quad (3.23)$$

We choose the parameter a , according to the property (v), in such a way that

$$\|x \log r_{n,a}\|_{\infty} < \frac{\eta}{2^n} \quad (3.24)$$

is satisfied. For brevity we denote r_n instead of $r_{n,a}$, henceforth.

The properties (iii) and (iv) imply that there exists a neighborhood W_n of u_n in the one point compactification of R such that $W_n \cap X = \emptyset$ and $|r_n(j\omega)| < M-1$, $\omega \in W_n$. (Note that a neighborhood of ∞ is $\{\omega \mid |\omega| > \Omega\}$ for some $\Omega > 0$.) U is compact since it is a closed subset of a compact set, and hence the cover $U \subset \bigcup_{n=1}^{\infty} W_n$ has a finite subcover, say $U \subset W = \bigcup_{n \in N} W_n$, where N is a finite index set.

Since P and r_n are outer

$$P(s) = \lambda \exp \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \log |P(j\omega)| \frac{\omega s + j}{\omega + js} \frac{d\omega}{1+\omega^2} \right] \quad (3.25)$$

and

$$r_n(s) = \lambda_n \exp \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \log |r_n(j\omega)| \frac{\omega s + j}{\omega + js} \frac{d\omega}{1+\omega^2} \right] \quad (3.26)$$

for $\lambda, \lambda_n \in \mathbb{C}$, $|\lambda| = |\lambda_n| = 1$.

Given $\delta > 0$, let

$$D_\delta = \{\omega \mid |P(j\omega)| \leq \delta\} \quad (3.27)$$

and, define $h_\delta(s)$ by

$$h_\delta(s) = \bar{\lambda} \prod_{n \in \mathbb{N}} \lambda_n \exp \left[\frac{1}{\pi} \int_{-\infty}^{\infty} [-C_\delta(\omega)] \frac{\omega s + j}{\omega + js} \frac{d\omega}{1+\omega^2} \right] \quad (3.28)$$

where

$$C_\delta(\omega) = \begin{cases} 0 & \omega \in D_\delta \\ C(\omega) & \omega \notin D_\delta \end{cases} \quad (3.29)$$

$$C(\omega) = \log |P(j\omega)| - \sum_{n \in N} \log |r_n(j\omega)| \quad (3.30)$$

The proof will be completed if we show that $h_\delta \in H^\infty$ and that for sufficiently small $\delta > 0$ this function satisfies (3.19). Indeed, C_δ is uniformly bounded since $\log |P(j\omega)|$ and $\log |r_n|$ are bounded in ${}^cD_\delta$ (the complement of D_δ). Hence $h_\delta \in H^\infty$.

In verifying (3.19) we use the following equalities.

$$\begin{aligned} Ph_\delta(s) &= \prod_{n \in N} \lambda_n \exp \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \sum_{n \in N} |r_n(j\omega)| \frac{\omega s + j}{\omega + js} \frac{d\omega}{1+\omega^2} \right. \\ &\quad \left. + \frac{1}{\pi} \int_{D_\delta} C(\omega) \frac{\omega s + j}{\omega + js} \frac{d\omega}{1+\omega^2} \right] \\ &= \prod_{n \in N} r_n(s) \exp \left[\frac{1}{\pi} \int_{D_\delta} C(\omega) \frac{\omega s + j}{\omega + js} \frac{d\omega}{1+\omega^2} \right] \end{aligned} \quad (3.31)$$

and

$$|Ph_\delta(j\omega)| = \begin{cases} \prod_{n \in N} |r_n(j\omega)| & \omega \in D_\delta \\ |P(j\omega)| & \omega \notin D_\delta \end{cases} \quad (3.32)$$

which follow from (3.25), (3.29) and (3.30).

Note now that if $\omega \in W$ then $|r_n(j\omega)| < M-1$ for some index $n \in N$, and

$|r_n(j\omega)| \leq 1$, for all $n \in N$. So for $\delta < M-1$ there holds

$$|Ph_\delta(j\omega)| = \begin{cases} < M-1 & \omega \in W \\ \leq 1 & \omega \notin W \end{cases} \quad (3.33)$$

Consequently, for $\omega \in W$

$$|1 - Ph_\delta(j\omega)| \leq 1 + M-1 = M \quad (3.34)$$

For handling ω in the complement of W , we first observe that

$$\left| \arg \prod_{n \in N} r_n(j\omega) \right| \leq \sum_{n \in N} |\arg r_n(j\omega)| < \sum_{n=1}^{\infty} \pi 2^{-(n+2)} = \frac{\pi}{4} \quad (3.35)$$

from the property (vi) of r_n . Thus from Claim 8 below, for sufficiently small δ ,

$$|\arg Ph_\delta(j\omega)| < \frac{\pi}{4} \quad (3.36)$$

Hence (from (3.33), (3.36)) for $\omega \in W$,

$$|1 - Ph_\delta(j\omega)| \leq 1 < M. \quad (3.37)$$

Eq. (3.34) and (3.37) imply $\|1 - Ph_\delta\|_\infty < M$.

Finally we consider $\omega \notin W$; then

$$\begin{aligned}
|\log \prod_{n \in N} r_n(j\omega)| &\leq \sum_{n \in N} |\log r_n(j\omega)| \\
&\leq \sum_{n \in N} \frac{\eta}{2^n} = \eta \\
&< \sum_{n=1}^{\infty} \frac{\eta}{2^n} = \eta
\end{aligned} \tag{3.38}$$

by (3.24). From Claim 8 below it follows that for sufficiently small δ , we have

$$|\log Ph_\delta(j\omega)| < \eta \tag{3.39}$$

for $\omega \in X$. Thus Eqs. (3.23), and, (3.39) imply $\|x(1-Ph_\delta)\|_\infty < \varepsilon$, as required.

Q.E.D.

Claim 8

$$Ph_\delta \rightarrow \prod_{n \in N} r_n \text{ as } \delta \rightarrow 0 \text{ in } L^\infty(C_W)$$

Proof. Notice that the continuity of P implies $\bar{E}_\delta \cap C_W = \emptyset$ for small δ . Given that δ is indeed small, we have (from (3.32))

$$|Ph_\delta(j\omega)| = \prod_{n \in N} |r_n(j\omega)| \quad \text{for } \omega \in {}^c W. \quad (3.40)$$

Hence it remains to check that

$$\arg Ph_\delta(j\omega) \rightarrow \arg \prod_{n \in N} r_n(j\omega) \quad \text{as } \delta \rightarrow 0 \quad (3.41)$$

in $L^\infty({}^c W)$.

From (3.30), it suffices to show that

$$\int_{D_\delta} C(\omega) \frac{\omega\theta+1}{\omega-\theta} \frac{d\omega}{1+\omega^2} \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \quad (3.42)$$

uniformly for θ in ${}^c W$.

Since D_δ lies strictly within the interior of W , the kernel $(\omega\theta+1)/(\omega-\theta)$ is uniformly bounded over the domain $\omega \in D_\delta$ and $\theta \in {}^c W$. Setting $d\mu(\omega) = d\omega/(\omega^2+1)$, we know that $\log |P(\cdot)|$ and $C(\cdot)$ belong to $L^1(d\mu)$. By the first fact it is necessary that $\mu(D_\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Consequently, the second implies (3.42). This proves the claim. Q.E.D.

Remark. (i) Notice that we did not require continuity of $P(j\omega)$ at $\omega = \pm\infty$. In fact, the assumptions on the continuity of $P(j\omega)$ and the compactness of the subset X , can be relaxed in various ways without requiring considerable changes in the analysis. The current setup was chosen for simplicity.

(ii) A major part of the proof of Theorem 2 is dedicated to the construction of the "roll-off" functions r_n , which are needed when $2 \geq M >$

1. For $M > 2$, the assumptions can be further relaxed; e.g., if $P(j\omega)$ is continuous, to the requirement that P be outer and have no zeros in X .

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